Fractals

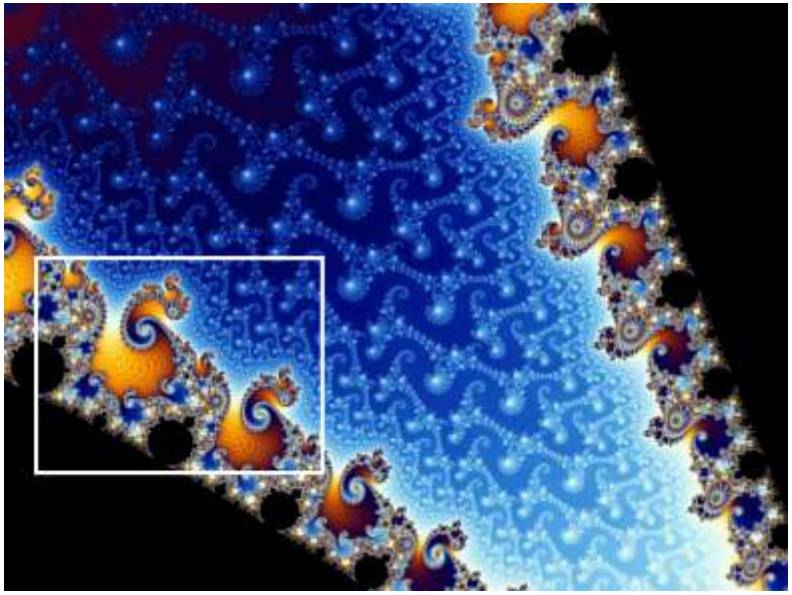
A tribute to Benoît Mandelbrot

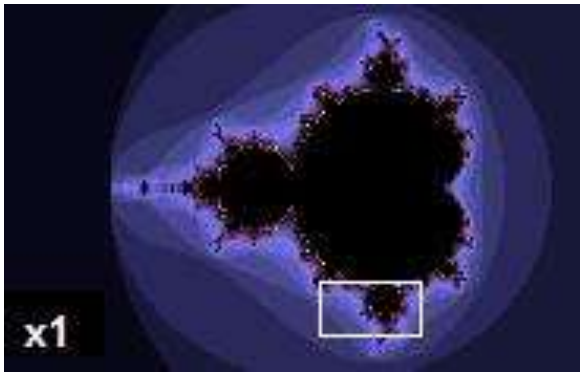
(1924-2010)

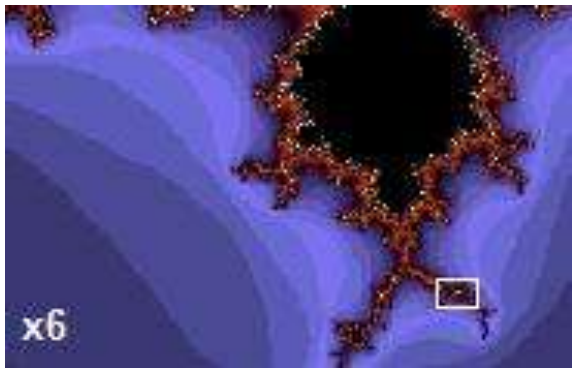
Math Club 11/01/2010

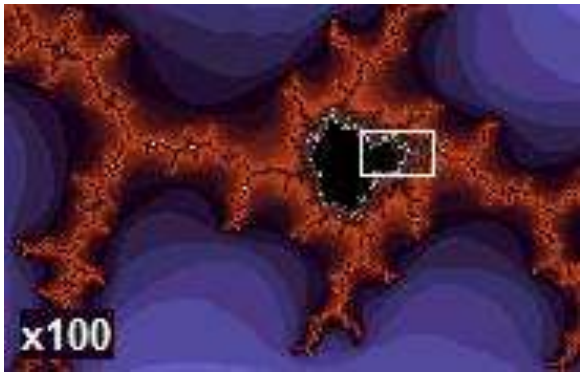
What is a fractal?

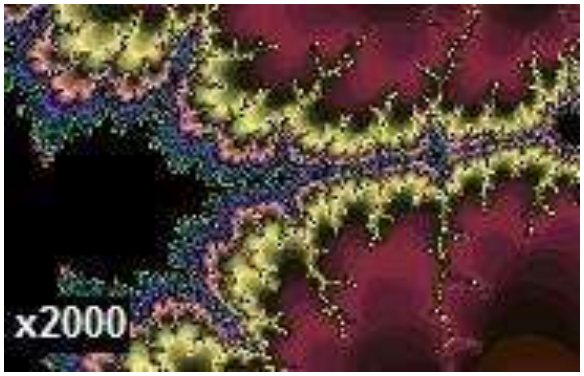
- A fractal is a shape that is self-similar (small parts of a fractal look like the whole fractal)
- Usually a fractal has finite area but infinite perimeter







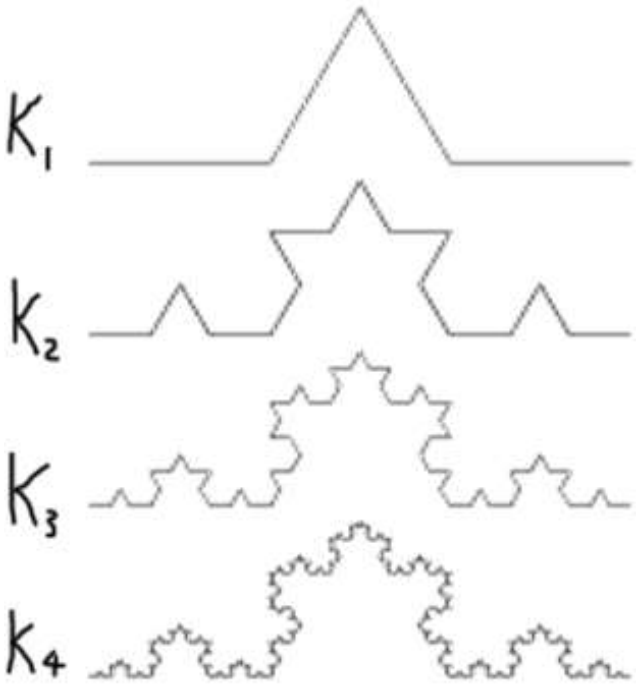




Mandelbrot Set

Koch Curve

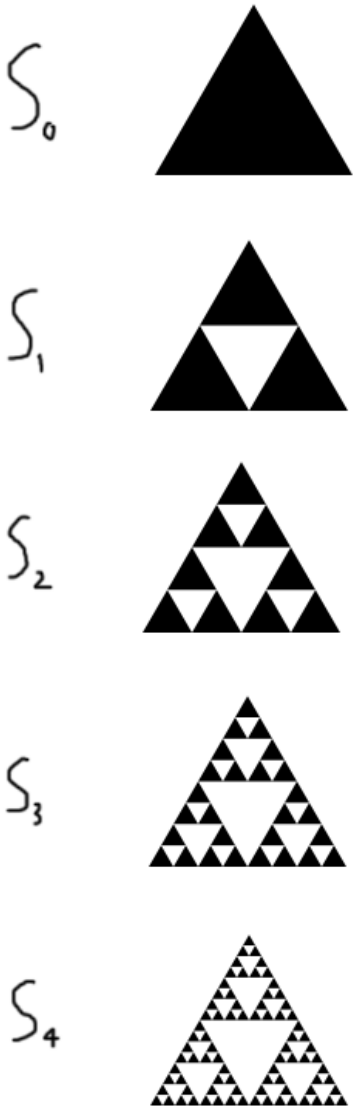
- A simple problem:
  - How many line segments are in K_n?
  - K_0 has 1 segment
  - K_1 has 4 segments
  - K_2 has 16 segments
  - K_{n+1} has 4 times as many as K_n
  - Therefore K_n has 4^n segments.
- Another simple problem:
  - What is the length of K_n?
  - K_0 is 1 unit long
  - A segment in K_{n+1} is 1/3 as long as one in K_n.
  - Therefore a segment in K^n has length $(1/3)^n$
  - K_n has 4^n segments
  - Length of $K_n = 4^n * (1/3)^n$
  - Which is equal to $(4/3)^n$.



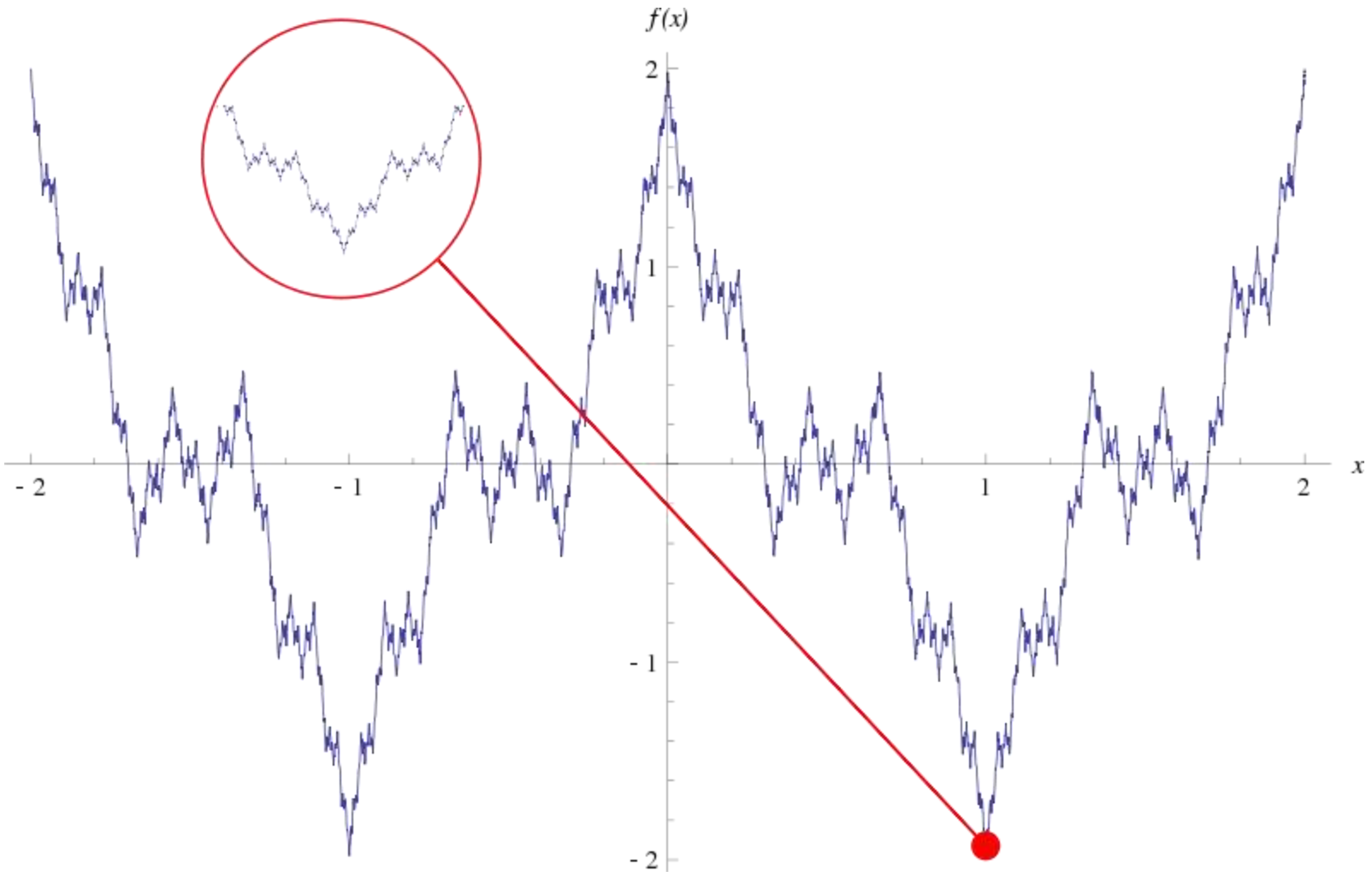
Sierpinski Triangle

- Problem:
  - How many black triangles are in S_n?
  - 1 triangle in S_0
  - 3 black triangles in S_1
  - S_{n+1} has 3 times more triangles than S_n
  - So S_n has 3^n triangles.

- Another problem:
  - What is the area of S_n?
  - S_0 has an area of 1
  - Each time, 1/4 of the area is removed
  - So area of S_{n+1} equals 3/4 * S_n
  - S_n has area of $(3/4)^n$



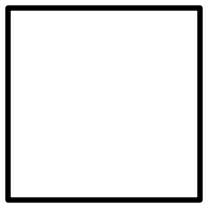
$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$



Weierstrass function: even functions can be fractals!

Dimensions

Dimension = 1: Line

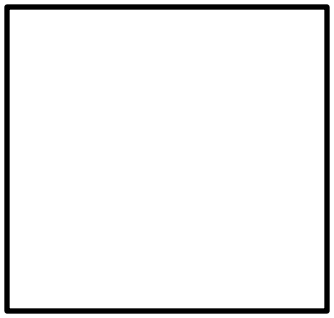


Dimension = 2: Square

Dimension = 3: Cube

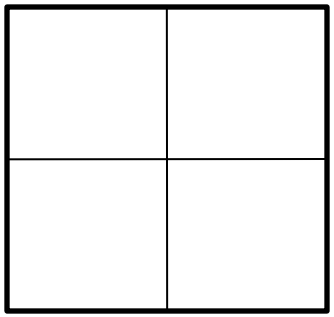
But why?

Why is a square 2-dimensional and a cube 3-dimensional?



1 square
Side length = 1

A square can be broken down into 4 identical pieces with a magnification of 2.



4 squares
Side length = 0.5

Now for cubes

1 cube
Side length = 1

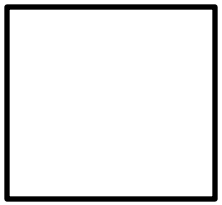
A cube can be broken down into 8 identical pieces, each with a magnification of 2.

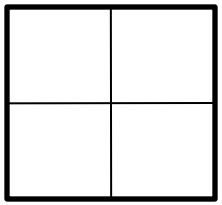
8 cubes
Side length = 0.5

Calculating Dimension

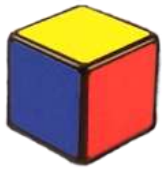
- Formula for dimension:

$$D = \frac{\log(\text{number of self similar objects})}{\log(\text{magnificat ion factor})}$$



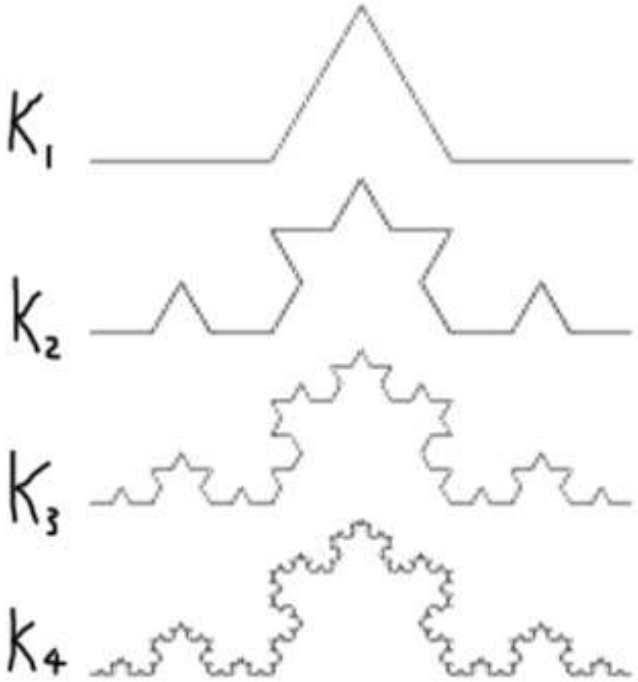


D = log(4)/log(2) = 2



D = log(8)/log(2) = 3

Koch Curve

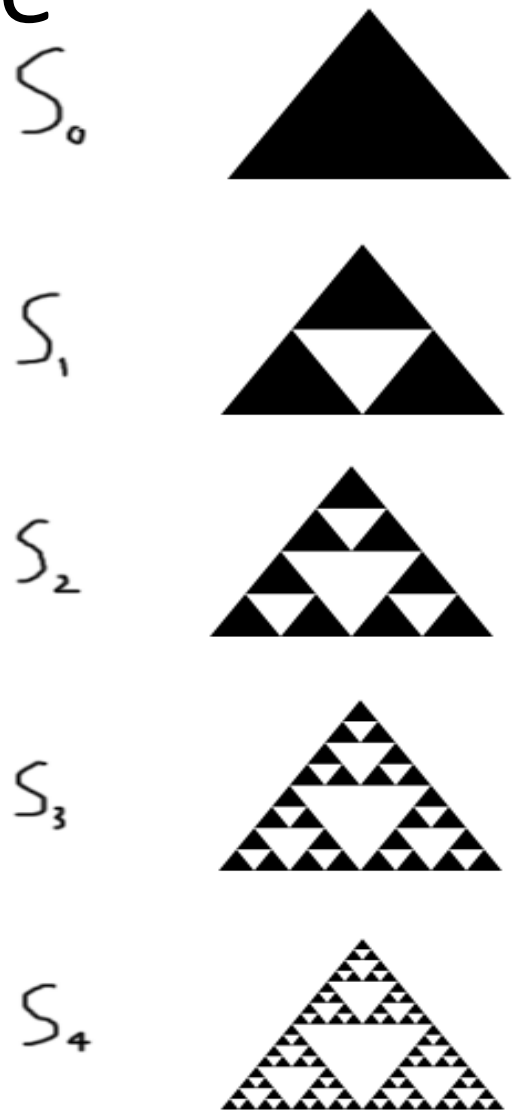


- What is the dimension?
- Each piece is split into 4 pieces
- Each piece is smaller by factor of 3
- Dimension = log(4)/log(3) ≈ 1.26

$$D = \frac{\log(\text{number of self similar objects})}{\log(\text{magnificat ion factor})}$$

Sierpinski Triangle

- What is the dimension?
- Each triangle becomes 3 smaller triangles
- Each triangle is smaller by factor of 2
- Dimension = log(3)/log(2) ≈ 1.58

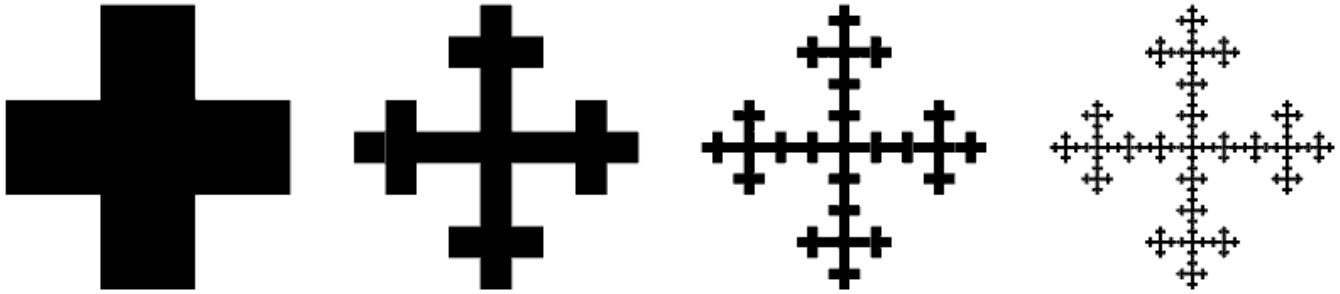


$$D = \frac{\log(\text{number of self similar objects})}{\log(\text{magnificat ion factor})}$$

Cantor Set

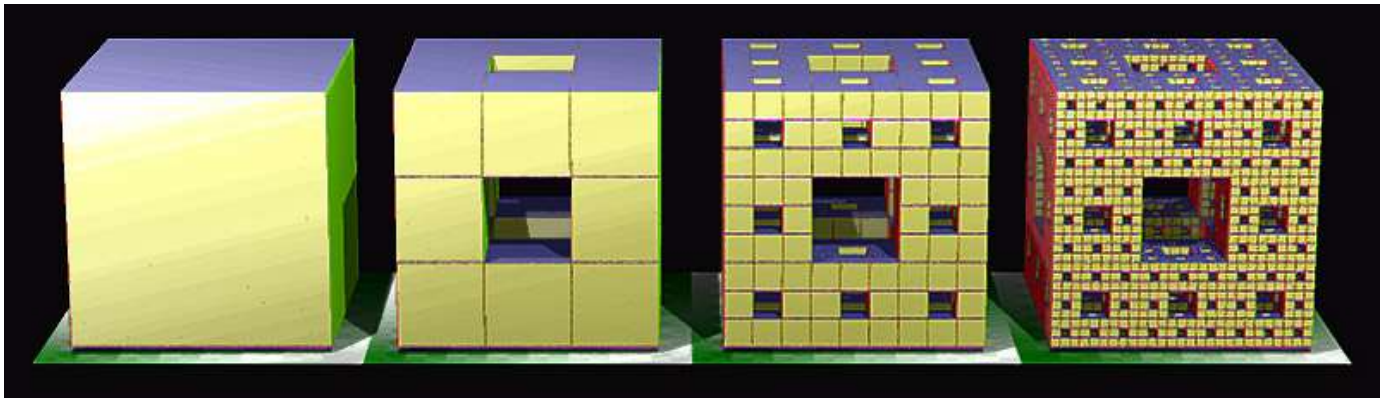
- What is the dimension?
- 2 pieces, each smaller by factor of 3
- Dimension = log(2)/log(3) ≈ 0.63

Vicsek Fractal



- What is the dimension?
- A cross becomes 5 crosses
- Each cross is 3 times smaller
- Dimension = log(5)/log(3) ≈ 1.46

Menger Sponge



- What is the dimension?
- Cube becomes 20 cubes, each 3 times smaller
- Dimension = log(20)/log(3) ≈ 2.73