Introduction to Functional Equations

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How do you solve them?

- A functional equation is an equation where the variable you want to find is a function.
- For example, suppose for every number x, $f\big(f(x)\big) = x$.
- Then $f(x) = -x$ is one solution.
- To fully solve a functional equation, we want to **find all solutions** to a functional equation, and prove that **no others** exist.

Really Easy Problem 1

- Find all functions f if $f(xy) = f(x) + f(y)$ for all reals.
- Answer: $f(x) = 0$.
- Let $y = 0$, then $f(0) = f(x) + f(0)$.
- So $f(x) = 0$.

Really Easy Problem 2

- Find all functions f such that $f(x+y) = f(x) + f(y)$ for all rational numbers. (Cauchy's Equation)
- Plugging $x = y = 0$ gives $f(0) = 0$.
- If we set $y = -x$, then $f(-x) = -f(x)$.
- If we set $y = x$, then $f(2x) = 2f(x)$; similarly $f(nx) = nf(x)$ by induction.
- Write $x = \frac{m}{n}$. Then $f(x \cdot n) = f(m \cdot 1)$ or $nf(x) = mf(1)$; $f(x) = \frac{m}{n} f(1)$.
- If $f(1) = c$, then the solution is $f(x) = cx$.

Really Easy Problem 3

- Find all functions f so that $f(\lfloor x \rfloor y) = f(x)\lfloor f(y) \rfloor$ for all reals. (IMO 2010)
- Put $x = y = 0$. Then either $f(0) = 0$ or $\lfloor f(0) \rfloor = 1$.
- Case 1: $\lfloor f(0) \rfloor = 1$. Putting $y = 0$, we get $f(x) = f(0)$, meaning the function is constant. Then for $[1,2)$ function $f(x) = c$ works.
- Case 2: if $f(0) = 0$. Putting $x = y = 1$ we get $f(1) = 0$ or $\lfloor f(1) \rfloor = 1$.
- Case 2a: if $f(1) = 0$, then putting $x = 1$ we get $f(y) = 0$, which is a solution.
- Case 2b: if $\lfloor f(1) \rfloor = 1$, putting $y = 1$ gives $f(\lfloor x \rfloor) = f(x)$; we now prove that this cannot be consistent with the rest of the problem.
- Putting $x = 2, y = \frac{1}{2}$ into the original, we get $f(1) = f(2)\left\lfloor f(\frac{1}{2}) \right\rfloor$.
- However if $f\left(\frac{1}{2}\right) = 0$ as the equation suggests, then $f(1) = 0$, a contridiction.
- So $f(x) = c$ for $x \epsilon [1,2)$ and $f(x) = 0$ otherwise.