Harmonic Conjugates

(And lots of circles)
(Because we really like circles)

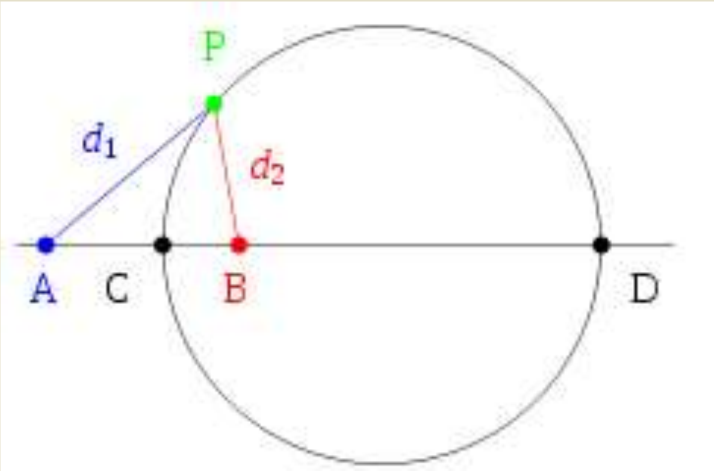
Math Club 12/05/2011

First Problem: Apollonius's Problem

- **Your ship at point A is chasing a ship on point B.**
- **Their ship is travelling in some direction at constant speed in a line.**
- **Your ship is k times faster than their ship.**
- **Your ship must also sail in a straight line at constant speed.**
- **Which way should you go?**

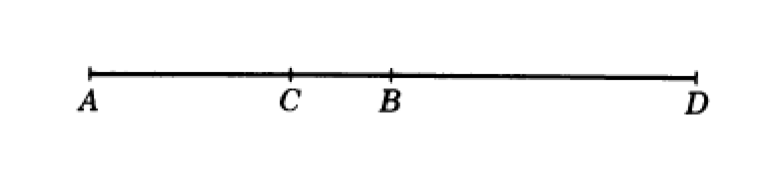
Second Problem: What is a circle?

- **Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι (*a circle is defined by a center and a radius*) – Euclid's Third Postulate**
- **Given A and B and a ratio k, a circle is the set of points P such that $\frac{PA}{PB} = k$ – Apollonius**



Harmonic Conjugates

- **Is it possible to divide a line segment internally and externally in the same ratio?**



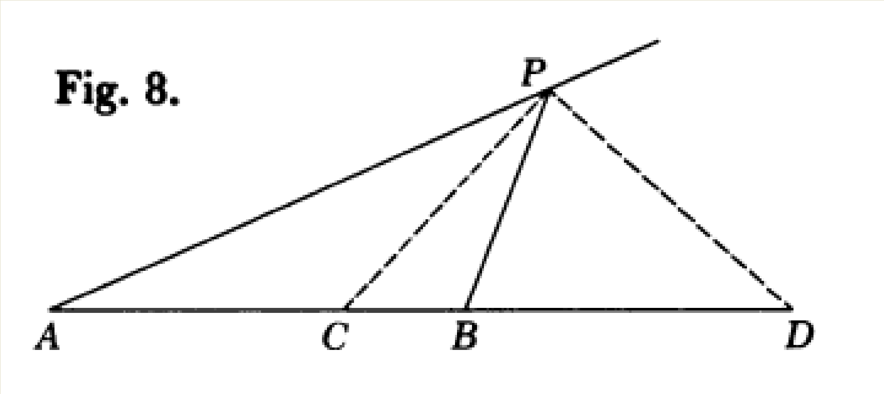
- **Here, let's say that** $\frac{AC}{BC} = \frac{AD}{BD}$.
- **We say C and D divide AB *harmonically*.**
- **Then notice that** $\frac{BC}{BD} = \frac{AC}{AD}$.
- **So A and B divide CD harmonically as well.**
- **AB and CD are *harmonic conjugates*.**

Constructing Harmonic Conjugates (1)

- We have points A and B fixed, and have in mind some number k.
- We want a point P such that AP is k times BP.
- Draw a circle of any length around B.
- Then draw a circle k times as long around A.
- Their intersection is a suitable P, if they intersect.

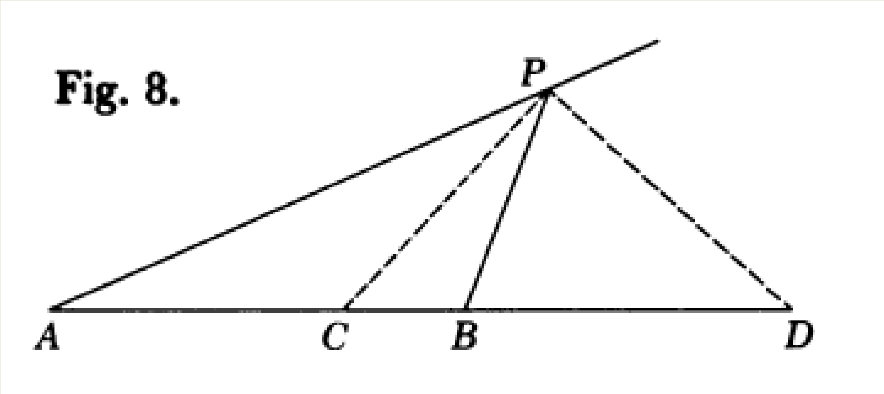
Constructing Harmonic Conjugates (2)

- We have a point P now.
- Draw the internal bisector of P and let C be the intersection with AB.
- Draw the external bisector of P and let D be the intersection.
- AB and CD are harmonic conjugates!



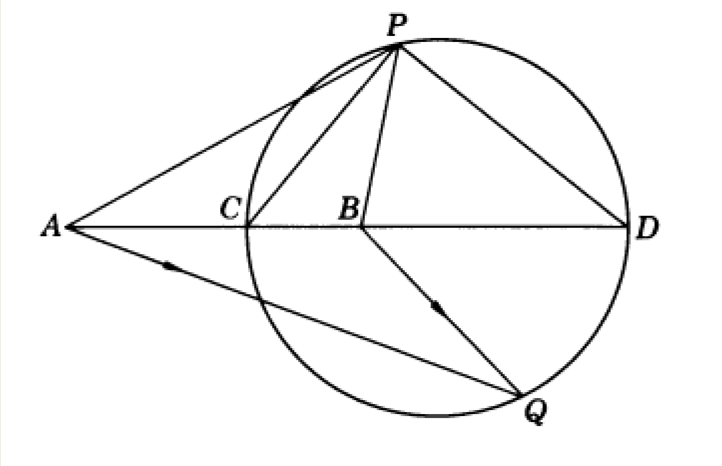
Constructing Harmonic Conjugates (3)

- Now does this actually work?
- By the angle bisector theorem, $\frac{AC}{BC} = \frac{AP}{BP} = k$.
- By the external angle bisector theorem, $\frac{AD}{BD} = \frac{AP}{BP} = k$.
- Hence $\frac{AC}{BC} = \frac{AD}{BD} = k$.
- So AB and CD are harmonic conjugates!



Constructing the Circle of Apollonius

- $\frac{AC}{BC} = \frac{AD}{BD} = k$
- **Notice that P is not part of this, so this works no matter what P we choose.**
- **Also notice that CPD is a right angle.**
- **Then P has to lie on the circle with diameter CD!**



Apollonius, we solved your problem

- Which direction do we need to go?
- Knowing AB, we construct the harmonic conjugates CD
- Then we draw the circle with diameter CD
- We know their direction, so they are going to intersect the circle at some point, say Q.
- We just sail towards Q!