



**Introduction to  
Multiplicative Functions**  
*Math Club 4/2/2012*

# What's the point?

- **How many divisors are there of 6800000?**
- **What's the sum of divisors of 6800000?**
- **What are the last five digits of  $21^{6800000}$ ?**
- ...
- **With multiplicative functions, we can solve these problems, and many many more!**

# What is a multiplicative function?

- A multiplicative function is an integer function where for all coprime  $m, n$ ,  $f(mn) = f(m)f(n)$
- (note: this might not work if  $m$  and  $n$  share some common divisors)
- For instance, the identity function  $f(a) = a$  is multiplicative.

# Multiplicative Function #1



- Let  $d(n)$  be the number of divisors of  $n$ .
- Then  $d(n)$  is a multiplicative function.
- Do you see why?

# Multiplicative Function #1



- **Consider the number  $5^3 \times 7^2$ .**
- **The divisors of  $5^3$  are  $1, 5, 5^2, 5^3$ .**
- **The divisors of  $7^2$  are  $1, 7, 7^2$ .**
- **If you multiply a number in the first list with one in the second list, you will get a number that divides into  $5^3 \times 7^2$ .**
- **Hence the number has 12 divisors.**

# Easy Problem #1

- How many divisors of 6800000 are there?

# Multiplicative Function #2



- Let  $\sigma(n)$  be the sum of divisors of  $n$ .
- Then  $\sigma(n)$  is multiplicative. (we're not going to prove this)

# Easy Problem #2



- **What is the sum of the 96 divisors of 6800000?**



# Multiplicative Function #3



- Let  $\phi(n)$  be the number of integers in the range  $[1, 2, \dots, n]$  that don't share any factors with  $n$ .
- For instance,  $\phi(12) = 4$  because **1, 5, 7, 11** don't have any factors in common with 12, but everything else does.
- The function  $\phi(n)$  is multiplicative (not going to prove this today)

# Practice with Phi

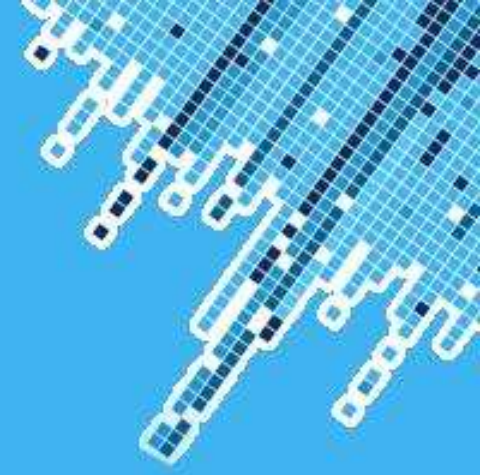
- Calculate  $\phi(13^8)$ .
- Calculate  $\phi(6800000)$ .

# Euler's Theorem



- **If  $a$  and  $n$  don't have any factors in common, then  $a^{\phi(n)} \equiv 1 \pmod{n}$**
- **For instance, since  $\phi(12) = 4$ , we have  $5^4$  has a remainder of 1 when divided by 12.**

# Easy Problem #3



- Find the last five digits of  $21^{6800000}$
- Hint: We want to find  $21^{6800000} \bmod 10^6$ .
- Hint: Use Euler's theorem:  $21^{\phi(10^6)} \equiv 1 \pmod{10^6}$