Introduction to Multiplicative Functions

Math Club 4/2/2012

What’s the point?

- **How many divisors are there of 6800000?**
- **What’s the sum of divisors of 6800000?**
- **What are the last five digits of $21^{6800000}$?**
- **…**
- **With multiplicative functions, we can solve these problems, and many many more!**

What is a multiplicative function?

- **A multiplicative function is an integer function where for all coprime m, n, $f(mn) = f(m)f(n)$**
- **(note: this might not work if m and n share some common divisors)**
- **For instance, the identity function $f(a) = a$ is multiplicative.**

Multiplicative Function #1

- **Let $d(n)$ be the number of divisors of n.**
- **Then $d(n)$ is a multiplicative function.**
- **Do you see why?**

Multiplicative Function #1

- **Consider the number $5^3 \times 7^2$.**
- **The divisors of 5^3 are $1, 5, 5^2, 5^3$.**
- **The divisors of 7^2 are $1, 7, 7^2$.**
- **If you multiply a number in the first list with one in the second list, you will get a number that divides into $5^3 \times 7^2$.**
- **Hence the number has 12 divisors.**

Easy Problem #1

- **How many divisors of 6800000 are there?**

Multiplicative Function #2

- **Let $\sigma(n)$ be the sum of divisors of n.**
- **Then $\sigma(n)$ is multiplicative. (we're not going to prove this)**

Easy Problem #2

- **What is the sum of the 96 divisors of 6800000?**

Multiplicative Function #3

- **Let $\phi(n)$ be the number of integers in the range [1,2,…,n] that don't share any factors with n.**
- **For instance, $\phi(12) = 4$ because 1,5,7,11 don't have any factors in common with 12, but everything else does.**
- **The function $\phi(n)$ is multiplicative (not going to prove this today)**

Practice with Phi

- **Calculate $\phi(13^8)$.**
- **Calculate $\phi(6800000)$.**

Euler’s Theorem

- **If a and n don’t have any factors in common, then $a^{\phi(n)} \equiv 1 \ (mod\, n)$**

- **For instance, since $\phi(12) = 4$, we have 5^4 has a remainder of 1 when divided by 12.**

Easy Problem #3

- **Find the last five digits of** $21^{6800000}$
- **Hint: We want to find** $21^{6800000}$ **mod** 10^6**.**
- **Hint: Use Euler's theorem:** $21^{\phi(10^6)} \equiv 1 \, mod \, 10^6$