Problem-Solving Techniques (part one)

HWW Math Club Meeting (April 4, 2011)

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NO CALCULATORS.

Intro: Cool number, bro

In this problem we deduce certain properties of the number 314159265.

1. Prove that $45 \mid 314159265$.
2. Calculate $\lceil \log_{10}(314159265) \rceil$.
3. What is the 314159265^{th} derivative of $\sin(x)$?

Approach 1: Teakettle Principle

- Reduce an unknown problem to one you already know how to do (well, duh).
- Solve

$$8x^{0.73} - x^{0.48} - 69x^{0.23} = 0$$

$$\Rightarrow x^{0.23}(8x^{0.5} - x^{0.25} - 69) = 0$$

$$(let\ u = x^{0.25}) \Rightarrow x^{0.23}(8u^2 - u - 69) = 0$$

$$\Rightarrow -x^{0.23}(8u + 23)(u - 3) = 0$$

$x = 0$ *and* $x = 81$ are solutions.

Protip: Graphing is Cool

- How many real solutions are there to the equation:

$$x^{100} - 4^x * x^{98} - x^2 + 4^x = 0?$$

$$\Rightarrow x^{98}(x^2 - 4^x) - (x^2 - 4^x) = 0$$
$$\Rightarrow (x^{98} - 1)(x^2 - 4^x) = 0$$

$x^{98} - 1 = 0$ gives two real solutions, $x = \pm 1$.
$x^2 - 4^x$ factored gives $(x - 2^x)(x + 2^x)$, not very helpful, so we graph!

$\therefore$ There are three real solutions. ■

Approach 2:
Work Backwards, Write Forwards

Prove the two-variable AM-GM inequality:

$$\frac{a+b}{2} \geq \sqrt{ab} \; \forall \; a, b \geq 0$$

Proof:

$$(a-b)^2 \geq 0$$
$$\Rightarrow a^2 - 2ab + b^2 \geq 0$$
$$\Rightarrow a^2 + 2ab + b^2 = (a+b)^2 \geq 4ab$$
$$\Rightarrow a + b \geq 2\sqrt{ab}$$
$$\Rightarrow \frac{a+b}{2} \geq \sqrt{ab}. \blacksquare$$

PROTIP: WHEN ALL ELSE FAILS…

- (Hypatia 2007) Cities A,B,C,D,E are all connected to each other. Penny starts at A, visits all cities, then returns to A. If her route is of the form A→__→__→E→__→A, how many possible routes are there?
- Solution (this is an official one!):
- List the possibilities: A→B→C→E→D→A, A→B→D→E→C→A, A→C→B→E→D→A, A→C→D→E→B→A, A→D→B→E→C→A, A→D→C→E→B→A
- Therefore there are 6 possible routes. ■

Approach 3: Look Before you Leap

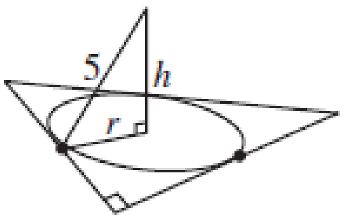
- Figure out what you want and how you're going to get it.
- (Euclid 2002): Three metal rods of lengths 9, 12, and 15 and negligible widths are welded to form a right-angled triangle held horizontally. A sphere of radius 5 sits on the triangle so that it is tangent to each rod. How high is the top of the sphere from the plane of triangle?

Approach 3: Look Before you Leap

- We want the height of sphere above the triangle
- Thus we want where on sphere at which the triangle is tangent to the sphere
- This would be the incircle (the unique circle inscribed in triangle)
- Tasks:
  - Find the incircle's unique radius (the inradius)
  - Find where along the sphere that radius occurs
  - Find height from there to the top of the sphere

Approach 3: Look Before you Leap

- $A = rs$
- $r_i = \frac{A}{s} = \frac{bh}{2s} = \frac{(9)(12)}{2\left(\frac{9+12+15}{2}\right)} = 3$



- $r_s = 5, r_i = 3$ so $h = 4$ (to center of sphere)
- The other half of the circle has height = 5, so
- The top of the sphere is 9 units above the plane of the triangle.

TEH ENDS (FOR NAO)

Cool Inequality, Bro:

Prove $(x+y)(y+z)(x+z) \geq 8xyz$ for non-negative x, y, z.

Sample problem covered next time:

Let $S(n)$ be the digit sum function.

Find $S\left(S\left(S\left(S\left(4444^{4444^{4444}}\right)\right)\right)\right)$.